The Gyrokinetic Regime
Geometry
Velocity Space
Linear How-To
Units
Non-Linear Issues
Miscellany
Coordinates

Start from Vlasov equation with a collision operator:

\[
\frac{\partial F_s}{\partial t} + v \cdot \nabla F_s + a \cdot \frac{\partial F_s}{\partial v} = C(F_s)
\]

Acknowledge parallel and perpendicular dynamics are different:

\[
v = v_\perp + \hat{b} v_\parallel.
\]

Go to energy and magnetic moment coordinates

\[
E = \frac{1}{2}mv^2 \quad \mu = \frac{1}{2}mv_\perp^2/B
\]

with inverse transformation

\[
v_\parallel^2 = 2(E - \mu B)/m \quad v_\perp = v_\perp(e_1 \cos \xi + e_2 \sin \xi)
\]

where \(e_1, e_2, \hat{b}\) form an orthogonal coordinate system.
Guiding Center Transformation

- First key step is coordinate transformation, from
  \[(x, v_\parallel, v_\perp, \xi) \rightarrow (x', E, \mu, \xi)\]

- Second key step is average over \(\xi\)

- The difference between \(x\) and \(x'\) is the difference between the position of the particle and its guiding center.
  (Perpendicular velocity is subtle!)
Ordering Assumptions

- Define:
  \[
  \left| \frac{\rho}{F_0} \frac{\partial F_0}{\partial x} \right| \sim \frac{\rho}{L} \equiv \rho_*
  \]

- Require **slow** evolution of equilibrium:
  \[
  \left| \frac{1}{\Omega F_0} \frac{\partial F_0}{\partial t} \right| \sim \rho_*^3
  \]
  (transport time scale, \( \tau \))

- For fluctuations, require
  \[
  \frac{\omega}{\Omega} \sim \rho \hat{b} \cdot \nabla' \sim \frac{\delta f}{f} \sim \frac{\delta B}{B} \sim \frac{v_E}{v_t} \sim \rho_*
  \]
  but allow
  \[
  \rho \hat{b} \times \nabla' = k_\perp \rho \sim 1
  \]

- Note there are **three** time scales:
  \[
  \Omega^{-1}, \quad \omega^{-1}, \quad \tau
  \]
Dynamical Equation

- Expand $F$ and fields in small parameter ($\sim \rho_*$)

- Find equilibrium is independent of gyrophase $\xi$, $F_0 = F_0(E, \mu, x)$. Solubility condition yields $\hat{b} \cdot \nabla F_0 = 0$

- Assume equilibrium has isotropic pressure: $F_0 = F_0(E, x_\perp)$

- Perturbed distribution function still has $\xi$ dependence. GK equation describes evolution of $h$, the non-adiabatic, $\xi$-independent part:

\[
\left( \frac{d}{dt} + v_\parallel \hat{b} \cdot \nabla + i\omega_d + C \right) h = i\omega_* \chi - q \frac{\partial F_0}{\partial \epsilon} \frac{\partial \chi}{\partial t}.
\]
Notation Defined

- Time derivative includes nonlinear terms:
  \[
  \frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{c}{B} \{\chi, h\}.
  \]

- Generalized potential is
  \[
  \chi = J_0(\gamma) \left( \Phi - \frac{v_\parallel}{c} A_\parallel \right) + \frac{J_1(\gamma)}{\gamma} \frac{mv_\perp^2 B}{q B}.
  \]

- Argument of Bessel functions is \( \gamma = k_\perp v_\perp / \Omega \)

- Curvature and \( \nabla B \) drifts from \( \omega_d \):
  \[
  \omega_d = k_\perp \cdot B_0 \times \left( m v_\parallel^2 \hat{b} \cdot \nabla \hat{b} + \mu \nabla B_0 \right) / (m B_0 \Omega),
  \]
Integrals over Velocity

- To find fields, need Maxwell’s equations. Sources (charge, current) are integrals over velocity and sums over species.

- Integrals evaluated at position $x$; requires coordinate transformation for $h = h(E, \mu, x')$. Example:

$$\int d^3v \, h = \frac{B}{m^2} \int \frac{d\epsilon \, d\mu \, d\xi}{|v||} \, h \exp(iL) \equiv \frac{1}{2\pi} \int d^2v \, d\xi \, h \exp(iL)$$

where $L = (v \times \hat{b} \cdot k_\perp)/\Omega$ accounts for the gyrophase dependence.

- Integral over $\xi$ results in Bessel functions:

$$\frac{1}{2\pi} \int d\xi \, h \exp(iL) = hJ_0(\gamma)$$
Maxwell’s Equations

- Poisson’s equation:
  \[ \nabla^2_{\perp} \Phi = 4\pi \sum_s \int d^2v \, q \left[ q\Phi \frac{\partial F_0}{\partial E} + J_0(\gamma)h \right] \]

- Ampere’s law:
  \[ \nabla^2_{\perp} A_{\parallel} = -\frac{4\pi}{c} \sum_s \int d^2v \, q v_{\parallel} J_0(\gamma)h \]

- Perturbed force balance:
  \[ \frac{\delta B_{\parallel}}{B} = -\frac{4\pi}{B^2} \sum_s \int d^2v \, mv_{\perp}^2 \frac{J_1(\gamma)}{\gamma} h \]
Review of Key Points

- Existence of multiple space and time scales:
  - Dynamics slow compared to $\Omega$
  - Equilibrium frozen on dynamical time scale
  - Weak variation of equilibrium scale lengths

- Equilibrium quantities constant on flux surface

- Small amplitude fluctuations
Review of Key Points

- Velocity-space coordinates are \((E, \mu)\)
  - Can trade magnetic moment \(\mu\) for pitch angle \(\lambda = \mu/E\)

- \(k_{||} \ll k_{\perp}\) implies high toroidal mode numbers.

- No restriction on any of
  \[
  \beta, \quad k_{\perp}\rho, \quad \frac{\omega}{k_{||}v_t}, \quad \frac{\omega}{\omega_d}, \quad \frac{\omega}{\omega_b}, \quad \frac{\omega}{\nu}, \quad \frac{\omega}{\omega_{NL}}
  \]
Additional Nonlinearities

- “Parallel” nonlinearity ordered small:
  \[ \hat{b} \cdot \nabla \Phi \left( \frac{\partial \delta f}{\partial v_\parallel} \right) \sim \hat{b} \cdot \nabla \Phi \left( \frac{\delta f}{v_t} \right) \ll \hat{b} \cdot \nabla \Phi \left( \frac{\partial F_0}{\partial v_\parallel} \right) \sim \hat{b} \cdot \nabla \Phi \left( \frac{F_0}{v_t} \right) \]
  Likely a good assumption in fully developed turbulence

- Nonlinearities in Maxwell equations:
  \[ \delta (n \Phi) \sim n_0 (\delta \Phi) + (\delta n) \Phi_0 + (\delta n) (\delta \Phi) \]
  Dropped because fluctuation amplitudes are ordered small (no perp gradient here)

- Time evolution of equilibrium strictly forbidden
References

