Phys 762: Mid-term Examination

This is a take-home examination. From the time you begin to the time you finish, do not let more than two hours elapse. You may consult any printed or handwritten materials (books, notes from class, etc.). You may not use a computer (no internet searches, no Mathematica, etc.). You may use a hand calculator. Complete your work on the exam without consulting with anyone else. Don’t panic on Problem 1. I have tried to make it possible to get through it even if you can’t figure out some of the steps along the way.
Poloidal magnetic field is generated by four current-carrying bars located at $x = \pm a/\sqrt{2}, y = \pm a/\sqrt{2}$ as shown.
1. Particle orbits.

A “linear cusp” has an axial magnetic field $B_0 = B_0 \hat{z}$ and a poloidal magnetic field $B_{\text{pol}}$ (see figure) which is derived from the vector potential $A = A_z \hat{z}$, where

$$A_z = B_z \frac{xy}{a} \quad \text{and} \quad B_z = \frac{8I}{ca}.$$

(a) Use the given information to show that

$$B_{\text{pol}} \cdot \nabla A_z = 0.$$

This equation states that $A_z$ is constant along poloidal field lines. Put another way, the projection of a magnetic field line on the poloidal ($x,y$) plane is described by the equation $A_z = C$. You will use this information in the rest of this problem.

(b) Consider the field lines in the first quadrant. Find an expression for $A_z$ in polar coordinates and parameterize the lines by $r_0$, their distance of closest approach to the origin. I.e., express $C$ in terms of $r_0, B_z, \alpha$. Use this expression to show that the field lines are described by

$$r^2 = \frac{r_0^2}{2 \cos \theta \sin \theta}.$$

(c) Show that the magnitude of the total magnetic field is given by

$$B = \sqrt{B_0^2 + B_{\text{pol}}^2 \frac{\alpha^2}{a^2}}.$$

You may wish to recall that in cylindrical coordinates, the curl operator has components:

$$\nabla \times \mathbf{A} = \frac{1}{r} \frac{\partial (r A_\theta)}{\partial \phi} - \frac{\partial A_z}{\partial \phi} = \frac{1}{r} \frac{\partial r A_\phi}{\partial \theta} - \frac{1}{r} \frac{\partial A_r}{\partial \theta}.$$

(d) Using the results from (b) and (c), find a formula expressing the field magnitude $B$ along field line $r_0$ in terms of $B_0, B_{\text{pol}}, r_0, a, \alpha$. You do not need to find an explicit expression for $\theta_M$. Show that your expression is equivalent to

$$v_1(\theta) = v \frac{1}{1 - \sqrt{1 + \frac{B_0^2 r_0^2}{B_z^2 a^2} \frac{1}{\sin 2\theta}}},$$

where

$$v_1(\theta) = v \frac{1}{1 - \sqrt{1 + \frac{B_0^2 r_0^2}{B_z^2 a^2} \frac{1}{\sin 2\theta_M}}}.$$

(f) You now have figured out that the particle trajectory in a linear cusp projected into the $x - y$ plane consists of a bounce motion. Taking into account what you know about the $\nabla B$ and curvature drifts, sketch this projected trajectory for a particle confined to the first quadrant, whose gyroradius is small compared to $r_0$. If you want to consider the case in which $B_0/B = 1$, you may (but please clearly state whatever assumptions you make). Explain the key features of your sketch in qualitative terms.
(g) This problem is translationally invariant in the $z$ direction. Consequently, the canonical momentum in the $z$ direction

$$p_z = mv_z + \frac{q}{c} A_z$$

must be conserved (i.e. independent of $\theta$). Use this fact, together with the relation

$$v_z = v_i \frac{B_0}{B}$$

(which you do not have to prove) to calculate an approximate formula for the width of the banana in your sketch. You may assume that the banana is skinny compared to $r_0$.

2. What is the speed, in centimeters per second, of a 1-eV electron? A 1-eV deuteron (1 proton + 1 neutron)? What is the cyclotron frequency in Hertz of a proton in a 5 Tesla field? A temperature of 1 eV is equivalent to what temperature in degrees Kelvin?

3. (a) Consider a plasma with a constant temperature $T_0$ in a straight magnetic field $\mathbf{B} = B_0 \mathbf{\hat{z}}$. If there is a density gradient $n'_0$ in the $x$-direction, use the MHD equilibrium equations to find an expression for the diamagnetic current.

(b) How can we reconcile the existence of this macroscopic current with the single particle motion in this system (where there are no perpendicular drifts)? Be as quantitative as you can.

4. Find an expression for the evolution of the entropy in an homogeneous, bounded plasma described by the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \nabla_v f = 0,$$

where the entropy is defined by

$$S = -\int d^3 x \int d^3 v f \ln f.$$

State clearly whatever assumptions you make.

5. Consider a more general, which we haven’t discussed. Take $G = G(f)$ and evaluate

$$\frac{d}{dt} \int d^3 x \int d^3 v G(f).$$

State your assumptions along the way. What do you conclude from this result?

6. The ideal MHD equations have the equation of state

$$p/\rho^\gamma = C,$$

where $C$ is a constant, as we discussed in class. The CGL equations (described in the handouts from class) have two equations of state, for $p_i$ and $p_i$, separately. Write down the CGL equations of state and describe (or show, if you have time) how to derive them from the moments of the drift kinetic equation.